Unit 4: Probability Distributions of Discrete Random Variables

MDM4U

Lesson Outline

Big Picture

Students will:

- understand probability distributions for discrete random variables;
- explore and connect binomial and hypergeometric distributions;
- recognize that the differences between a probability histogram and a frequency histogram may be the result of variability;
- complete a Games Fair culminating project.

Day	Lesson Title	Math Learning Goals	Expectations
1–2	Formalizing Discrete Random Variables Relating Probability Histograms and Frequency Histograms *(lessons not included)*	• Recognize and identify a discrete random variable. • Calculate the probabilities associated with all values of a random variable, with and without technology. • Generate, tabulate, and graph a probability distribution. • Compare a probability histogram to the frequency histogram of a related experiment conducted previously in the course. • Make connections between the frequency histogram and the probability histogram.	B1.1, B1.3
3	Spin and Win	• Calculate, interpret, and apply expected value. • Construct and analyse simple fair and unfair games, e.g., using familiar examples from Unit 1.	B1.2
4	Are You Game?	• Calculate, interpret, and apply expected value. • Make connections between the expected value and the weighted mean of the values of the discrete random variable. • Introduce the Games Fair project and its connection to experimental and theoretical probability. • Develop a critique for peer assessment for the Games Fair project.	B1.2 E1, E2,

Day	Lesson Title	Math Learning Goals	Expectations
5–7	The Binomial Distribution *(lessons not included)*	• Recognize and identify situations resulting in a binomial discrete random variable. • Calculate the probabilities associated with all values of a binomial random variable. • Compare binomial probability histograms to frequency histograms of related experiments conducted previously in the course. • Recognize that the differences between a probability histogram and a frequency histogram may be the result of variability and/or other factors (e.g., when tossing a coin 10 times and recording the number of heads, the frequency distribution should have some of the features of the probability distribution but will not match it exactly due to variability. As well, the model may not be correct due to bias such as an unfair coin or poor experimental technique or that independence is not a good assumption). • Investigate how increasing the number of independent trials makes the frequency histogram better match the probability histogram. • Generalize the algebraic representation of the binomial probability distribution. • Calculate, interpret, and apply expected value. • Compare the expected value of a binomial random variable to the mean of an experimental data set.	B1.1, B1.2, B1.3, B1.4, B1.6
8–9	The Hypergeometric Distribution *(lessons not included)*	• Recognize and identify situations resulting in a hypergeometric random variable. • Calculate the probabilities associated with all values of a random variable. • Compare hypergeometric probability histograms to frequency histograms of related experiments conducted previously in the course. • Calculate, interpret, and apply expected value. • Compare the expected value of a hypergeometric random variable to the mean of an experimental data set. • Investigate how changing the probability of "success" affects the shape of the probability histogram.	B1.1, B1.2, B1.3, B1.5, B1.6
10	Solve Problems *(lesson not included)*	• Solve problems involving uniform, binomial, and hypergeometric distributions. • Apply probability distributions to real-world situations. • Solve problems related to the design of a game for the Games Fair project (ensure that the expected value of the winnings falls within a given range).	B1.7 E1, E2
11–12	Games Fair Project data collection and presentation *(lessons not included)*	• Present games and collect data. • Organize the data. • Compare the probability model to the data gathered at the *Games Fair*. • Interpret, analyse, and summarize the data. • Communicate findings.	E1, E2
13–14	Jazz		

75 min

Math Learning Goals

- Calculate, interpret, and apply expected value.
- Construct and analyse simple fair and unfair games, e.g., using familiar examples from Unit 1.

Materials

- simulation manipulatives (e.g., numbered cubes, counters, graphing calculator)
- BLM 4.3.1, 4.3.2

Assessment Opportunities

Minds On…

Think/Pair/Share → Exploration

Use an acetate of BLM 4.3.1 to introduce the Spin and Win game.

Individual students complete the Anticipation Guide Before section (BLM 4.3.2.).

In pairs, students discuss the questions from the Anticipation Guide.

Pairs select from a variety of manipulatives and explore the game using a simulation of the game.

Engage students in a discussion of the difference of "chance" and "skill" highlighting that theoretical probabilities can be calculated for games of "chance" but not for games of "skill."

Playing the spinner game can be simulated, e.g., using a 6-sided numbered cube, a probability simulator.

Action!

Pairs → Experiment

Students determine experimentally the average winnings per game by:

- playing a specified number of trials, e.g., 25 trials, collecting data for each outcome, and determining average winnings per game;
- determining the average class winnings per game, showing calculations (Post some strategies so that students will know how to do this.)

Whole Group → Guided Discussion

Introduce the concept of expected value.

Students construct and complete the chart to determine the Expected Value of the Spin and Win Game:

Outcome	Theoretical Probability	Outcome × *P*(*Outcome Occurs*)

Summarize Expected Value [$E(x) = \sum(outcome \times P(outcome\ occures))$ or $E(x) = \sum(x \times P(X = x))$] and compare this value to the average winnings per game obtained experimentally.

Pairs → Explore

Students select What If… statements from a list provided by the teacher, e.g., What if …it costs $2 to play, is it a fair game? …the $5 prize increases to $6? …the $1 prize becomes "game over"? …the expected value must be between $1.40–$1.50?, and determine the effect on the expected value.

Process Expectations/Communicating/Observation: Observe pairs as they make changes to the game and connect results to expected value. Listen to discussions and ideas looking for items that students can share with others during the **Consolidate Debrief**.

Consolidate Debrief

Whole Class → Discussion

Develop a class definition of a Fair Game.

Individual → Self Assessment

Students complete the After section of the Anticipation Guide (BLM 4.3.2).

Application
Differentiated
Reflection

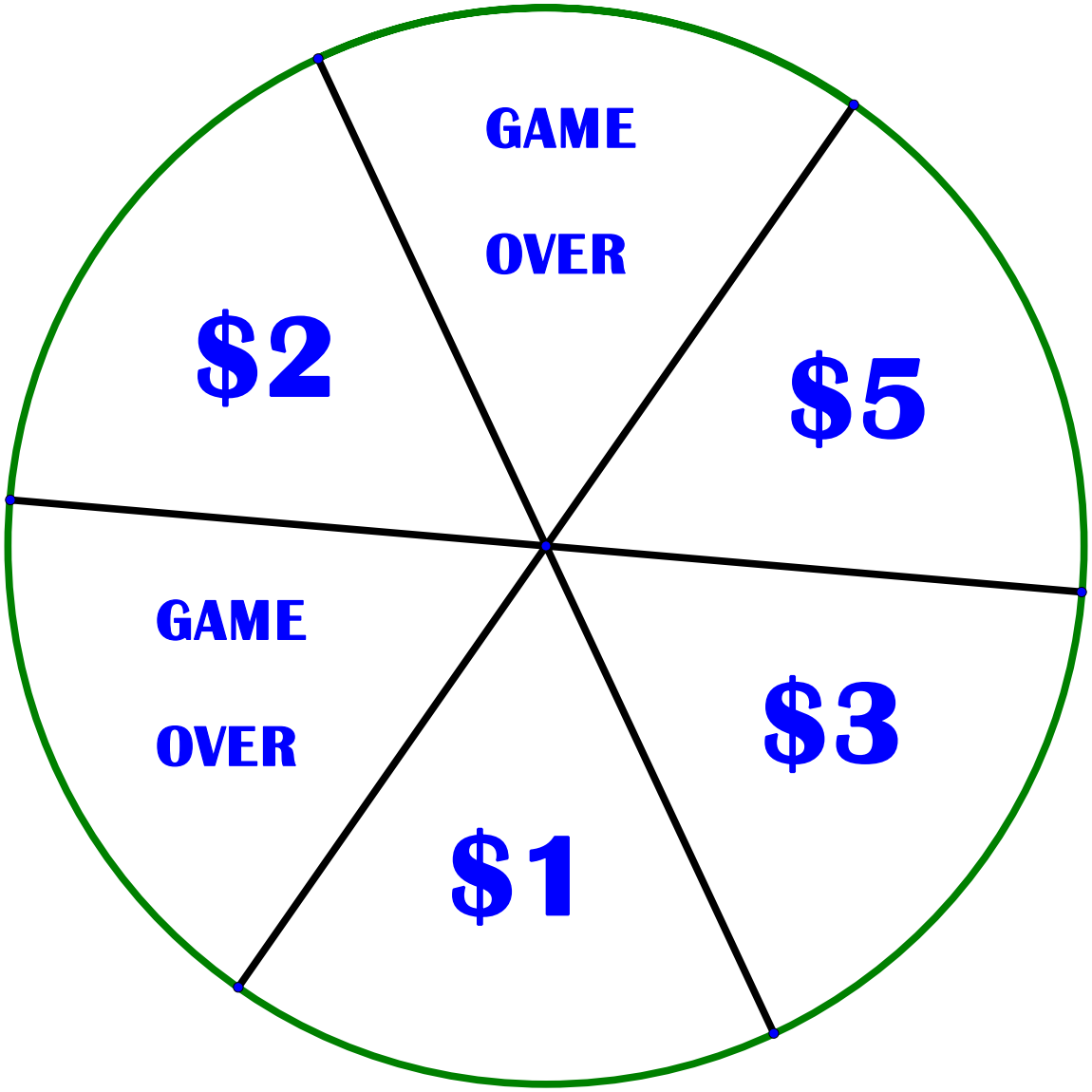
Home Activity or Further Classroom Consolidation

Create a new Spin and Win game that does not have 6 sections, determine the cost to play and the values for each outcome that will lead to an expected value in a set range. In your journal, discuss if the game is fair, with justification.

4.3.1: Spin and Win

Instructions

- one game costs $2 to play
- spin and win the amount shown



4.3.2: Anticipation Guide for Spin and Win

Instructions

- Check **Agree** or **Disagree**, in ink, in the **Before** category beside each statement before you start to play the Spin and Win game.
- Compare your choice with your partner.
- Revisit your choices at the end of the investigation.

Before		Statement	After	
Agree	**Disagree**		**Agree**	**Disagree**
		1. Playing this game requires skill.		
		2. You'll usually win when you play this game.		
		3. Every time you play this game, you have the same chance of losing.		
		4. This is a fair game.		

75 min

Math Learning Goals

- Calculate, interpret, and apply expected value.
- Make connections between the expected value and the weighted mean of the values of the discrete random variable.
- Introduce the Games Fair project and its connection to experimental and theoretical probability.
- Develop a critique for peer assessment for the Games Fair.

Materials

- coins, numbered cubes, cards
- BLM 4.4.1, 4.4.2, 4.4.3

Assessment Opportunities

Minds On…

Whole Class → Inside Outside Circle

Students discuss the strategies they used in the homework activity from Day 3.

Inside Outside Circle see TIPS for Teachers, TIP 13, p. 14

Whole Class → Discussion

Summarize and post the key strategies for altering the expected value of a game (e.g., change the dollar value or the probability of the outcome) and for determining the fairness of the game, e.g., the cost to play.

Action!

Whole Class → Activity Instruction

Introduce the Lucky Aces game using the following chart:

Winning Amount ($)	Roll of 6-Sided Numbered Cube	Coin Toss	Card Draw
2	1 or 2	Heads	Red
4	3 or 4	Tails	Spades
6	5 or 6	Heads	Ace

All other combinations result in $0. (The game involves rolling a numbered cube, flipping a coin and drawing a card from a standard deck).

Using a game in this way connects the concepts: tree diagrams, graphing probability distributions, and the expected value as they are related to the Games Fair project.

Pairs → Experiment

Students play the Lucky Aces game 20 times and collect data for the four possible outcomes in a tally chart.

Later in the unit, introduce examples of games involving binomial and hypergeometric distributions so that students have examples of these distributions to guide them.

Whole Class → Discussion

Students collate their results, record the experimental probabilities for each outcome, and construct the graph of the experimental probability distribution. Students calculate the theoretical probability of each outcome, using a probability tree diagram, graph the theoretical probability distribution, and calculate the Expected Value for this game.

Students dialogue about the differences between the experimental and theoretical probability distributions and explain their reasoning.

Learning Skills/Teamwork/Checkbric: Observe students as they play the game and record results. Note any areas of difficulty so they can be discussed during **Consolidate Debrief**.

Critique handouts could be provided to students ahead of the Games Fair. Feedback from peers may be used in the report to inform how the game might be improved.

Consolidate Debrief

Whole Class → Discussion

Introduce Games Fair (BLM 4.4.1 and 4.4.3).

Students assess the games Spin and Win BLM 4.3.1 and Lucky Aces using the criteria from Games Fair Assessment BLM 4.4.2 and discuss why Lucky Aces does not meet the criteria. Students identify any difficulties encountered in the Lucky Aces activity.

Brainstorm criteria for critiquing peers' games that will be used during the Games Fair, e.g., clear instructions, inviting display, appears to be within the expected value range. Agree on approximately five criteria that will be used for the critique.

Application
Exploration

Home Activity or Further Classroom Consolidation

Generate ideas for the design of your own game and produce an informal plan for a game for the Fair.

Students work on their game throughout the remainder of the unit.

4.4.1: Games Fair

To apply what you have learned about permutations, combinations, probability, and discrete probability distributions we are going to have a Games Fair! The high stakes Games Fair will take place over two classes.

Your task is to create an original, interesting, easy-to-play, and profitable game involving numbered cubes, spinners, cards, or any other reasonable item that introduces an element of chance. The game will require the participant to pay one or two 'Data Dollars' and all prizes must be payable in full 'Data Dollars.'

The expected value is dependent on the cost to play as identified in the chart:

Cost to Play One Game	Expected Value Range
$1	$0.60–$0.80
$2	$1.60–$1.80

On the first day of the Games Fair, half of the class are players, and the other half operate the games that they have created. On the second day, the roles are reversed. On Games Fair days, each player and each operator receives thirty 'Data Dollars.'

Operators

As an operator, create a tally chart for tracking all possible outcomes. If an operator or player no longer has enough 'Data Dollars' to award a prize or continue playing, bankruptcy is declared and $30 is borrowed from the ‘Data Bank.’ At the end of each day of Games Fair, the player and the operator with the most 'Data Dollars' are declared “winners.” Bankrupt players and/or operators are not eligible for a prize.

Players

On the day you are playing, you draw the names of three games. You must play each of these games a minimum of 5 times and provide a peer critique for each.
Each player must play every game at least twice.

After the Games Fair, you submit a report that includes:

- playing instructions and items required to play
- critiques from peers
- the theoretical probability of all possible game outcomes
- a graph of the theoretical distribution of all possible game outcomes
- a player's theoretical expected value
- a tally chart with the collected experimental results
- the experimental probability of all possible game outcomes
- a graph of the experimental distribution of all possible game outcomes
- a player’s experimental return per game

In the report, analyse, and provide reasons for differences between the experimental and theoretical results for all the components of your game; address feedback shared in the critiques and provide suggested modifications to make your game “better.” Make connections between your Game Fair experience and real world contexts.

4.4.2: Games Fair Assessment

Criteria	Level 1	Level 2	Level 3	Level 4
Problem Solving				
Appling mathematical processes and procedures correctly to determine accurate theoretical and experimental results for the components of the game	- correctly applies some of the mathematical processes and procedures with major errors	- correctly applies many of the mathematical processes and procedures with some errors	- correctly applies the mathematical processes and procedures with few errors	- correctly applies the mathematical processes and procedures with precision and accuracy
Connecting				
Relating the mathematical ideas of the Games Fair to real contexts	- makes weak connections between experimental and theoretical results to real contexts	- makes simple connections between experimental and theoretical results to real contexts	- makes appropriate connections between experimental and theoretical results to real contexts	- makes strong connections between experimental and theoretical results to real contexts
Reasoning and Proving				
Interpreting the mathematical results from the Games Fair and making relevant statements	- misinterprets a major part of the results from the games fair, but carries on to make some otherwise reasonable statements	- misinterprets a minor part of the results from the games fair, but carries on to make some otherwise reasonable statements	- correctly interprets the results from the games fair and makes reasonable statements	- correctly interprets the results from the games fair and makes insightful statements
Making inferences, conclusions and justifications when comparing theoretical and experimental results	- differences between theoretical and experimental results are not justified	- some differences between theoretical and experimental results are justified	- differences between theoretical and experimental results are justified	- differences between theoretical and experimental results are highly justified
Reflecting				
Using metacognitive strategies to suggest improvements for a better game	- applies metacognitive strategies to suggest improvements for a better game with limited insightfulness	- applies metacognitive strategies to suggest improvements for a better game with some insightfulness	- applies metacognitive strategies to suggest improvements for a better game with insightfulness	- applies metacognitive strategies to suggest improvements for a better game with a high degree of insightfulness
Communicating				
Integrating narrative and mathematical forms of communication in the report	- either mathematical or narrative form is present in the report, but not both	- both mathematical and narrative forms are present in the report, but the forms are not integrated	- both mathematical and narrative forms are present and integrated in the report	- a variety of mathematical and narrative forms are present, integrated in the report, and well chosen

4.4.2: Games Fair Assessment (continued)

The report includes:

- ☐ playing instructions, items required to play, and peer critiques
- ☐ theoretical probability of each possible game outcome
- ☐ a graph of the theoretical distribution of all possible game outcomes
- ☐ a player's theoretical expected value (within range)
- ☐ a tally chart with collected experimental results
- ☐ experimental probability of each possible game outcome
- ☐ a graph of the experimental distribution of all possible game outcomes
- ☐ a player's experimental return per game
- ☐ suggestions to improve the game
- ☐ connecting between the Games Fair and real contexts

4.4.3: Games Fair (Dollars)

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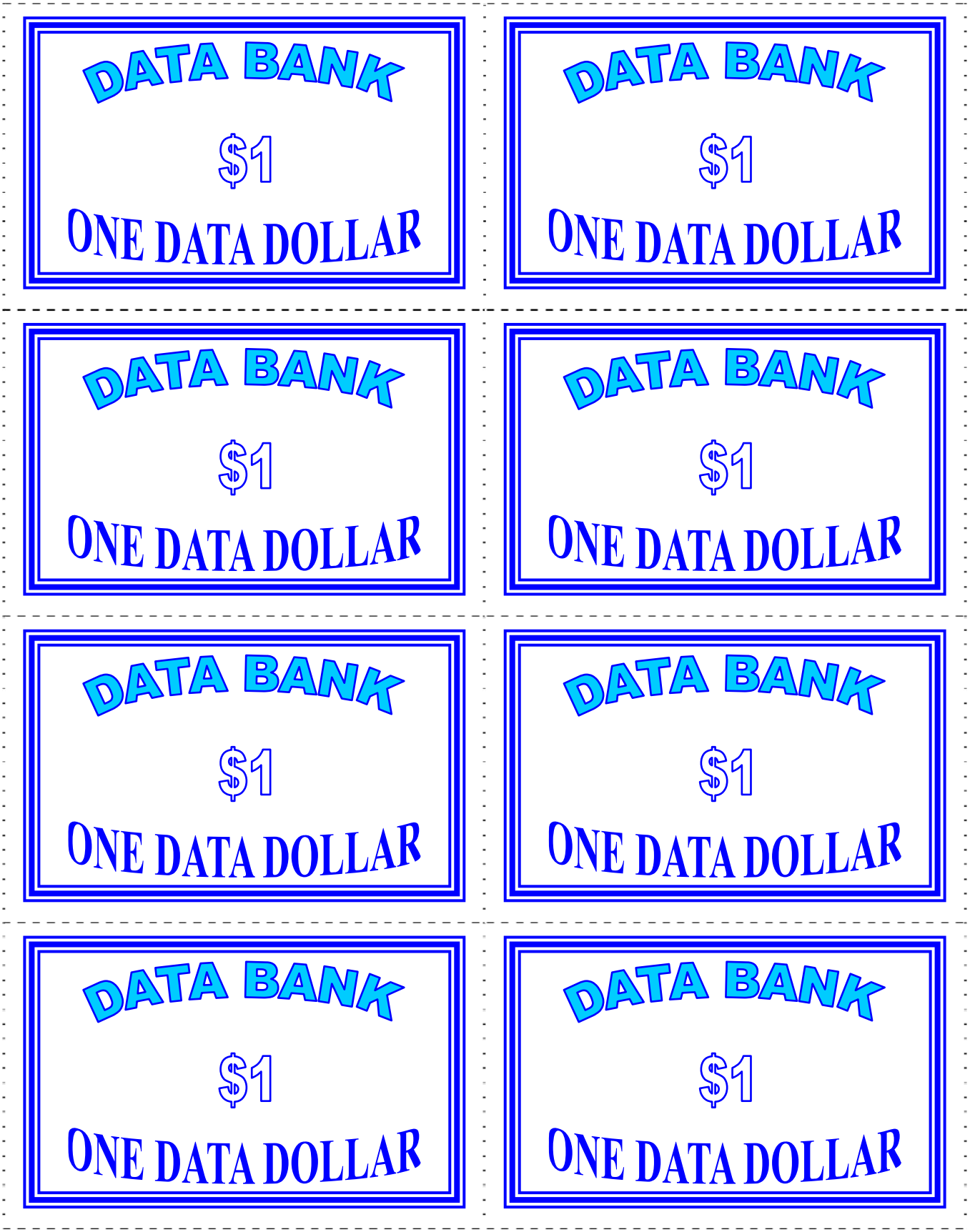

4.4.3: Games Fair (Dollars) (continued)

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